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STRESS APPROXIMATIONS BY THE h- AND p-VERSIONS OF THE FINITE ELEMENT METHOD

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March, 1982

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h- AND p-VERSIONS OF THE FINITE ELEMENT METHOD

B.A. Szabo* and Ivo Babuska**

ABSTRACT

The computational effort required for achieving a given level of precision in the root-mean-square measure of stress is examined. The main conclusion is that for most problems of practical importance it is not feasible to exercise substantive control of error in this norm by means of state of the art finite element technology. The computation and control of error of stress intensity factors is a much simpler problem.

1. Introduction

One of the commonly stated goals of finite element computations in structural analysis is to determine the state of stress in an elastic body. The results are often presented in the form of contour lines or surfaces connecting points at which a critical stress component or some combination of stress components has the same value. The contour lines or surfaces are produced by interpolating stress values computed at nodal points or Gauss points. The use of results generally falls into one or more of the following categories:

1. Identify the most highly stressed areas of a structure;

2. Determine how critical stress components change with respect to design modifications;

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3. Determine "nominal stresses" for the purpose of using experimentally established relationships between nominal stress and expected fatigue life for given structural details and given periodic loading parameters;

4. With increasing importance and frequency, direct evaluation of stress intensity factors associated with observed or hypothetical cracks at critical locations rather than determination of stress values is the principal goal of computations.

5. In problems involving material nonlinearity, the material properties are specified via equivalent stress-equivalent strain relationships. One of the goals of stress computations in such cases is to determine equivalent stress values at specific points, usually the Gauss points.

In engineering computations relative errors between 1 and 5 percent are generally deemed acceptable. 5 percent relative error ensures that the first digit is correct, in other words, the "accuracy" is one significant digit. 0.5 percent relative error ensures that the first two digits are correct.

In this paper we examine the relationship between relative error in the root-mean-square measure of stress and stress intensity factor values and the corresponding computational effort required in the h- and p-versions of the finite element method. We present examples based on problems in plane elasticity.

2. Convergence in energy

It is intuitively obvious that the effort required to obtain a given level of precision in some norm depends on the character of the function to be approximated. When a function does not have singularities or a large number of oscillations then the effort required for approximating it with piecewise polynomials is not substantial. In the solution

of plane elastic problems singularities and oscillations are caused by corners, loading, and sudden changes in boundary conditions or material properties. A typical corner detail is shown in Fig. 1.

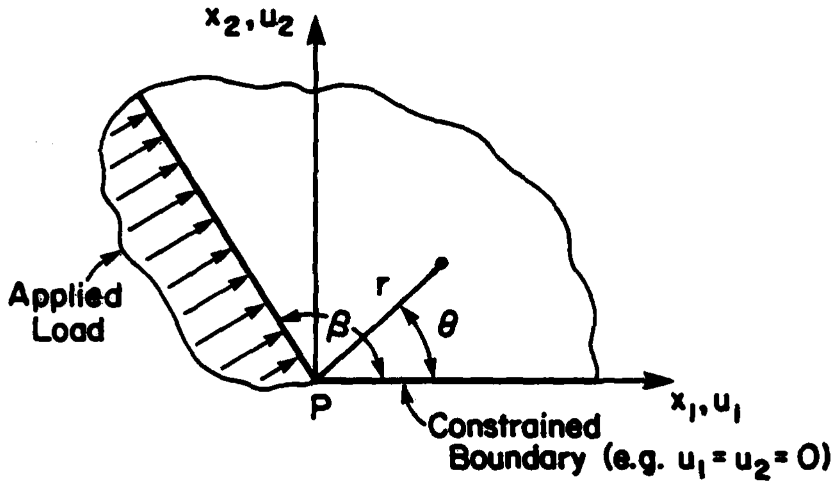

Fig. 1
Typical corner detail in plane elastic problems

In the immediate neighborhood of the corner the solution vector $\underline{u}$ can be written in terms of polar coordinates centered on the corner in the following form:

$$\underline{u} = r^{\alpha}\,\underline{F}(\theta) + \underline{G}(r,\theta) \tag{1}$$

In which $\underline{u} = \{u_1, u_2\}^T$, $\underline{F}$ and $\underline{G}$ are smoother functions than $r^{\alpha}F(\theta)$ and α is a positive number (in the case of homogeneous, isotropic solids $\alpha \geq \frac{1}{4}$) which depends on the angle β, the boundary conditions imposed on the sides meeting at P and Poisson's ratio ν.

The strains are obtained from the displacement field by differentiation

$$\underline{\varepsilon} = [D]\underline{u}(r,\theta) \tag{2}$$

in which:

$$\underline{\varepsilon} = \{\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{12}\}^T$$

$$[D] = \begin{bmatrix} \frac{\partial}{\partial x_1} & 0 \\ 0 & \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_1} \end{bmatrix}$$

and the stresses are obtained from the strains by multiplying $\underline{\varepsilon}$ with a symmetric, positive definite matrix [E]:

$$\underline{\sigma} = [E]\underline{\varepsilon} \tag{3}$$

The stresses are of the form:

$$\underline{\sigma} = r^{\alpha-1}\,\underline{\Phi}(\theta) + \underline{\psi}(r,\theta) \tag{4}$$

We note that for $\alpha < 1$, the stress is infinity at the corner. The values of α for various boundary conditions have been computed by Williams [1].

The strain energy per unit thickness (U) is defined as:

$$U(\underline{u}) = \frac{1}{2}\int_A \underline{\varepsilon}^T \underline{\sigma}\, d\, V = \frac{1}{2}\int_A ([D]\underline{u})^T [E]([D]\underline{u})\, dA \qquad (5)$$

The positive square root of $U(\underline{u})$ by definition is the <u>energy norm</u> of $\underline{u}$.

Most finite element formulations minimize the total potential energy which in the cases discussed here is equivalent to minimizing the strain energy of the error denoted by e_E^2:

$$e_E^2 = U(\underline{u}-\underline{u}_{FE}) = \left|U(\underline{u}) - U(\underline{u}_{FE})\right| \qquad (6)$$

in which $\underline{u}_{FE}$ denotes the finite element approximation to $\underline{u}$. Eq. (6) states the well known fact that the strain energy of the error equals the error in strain energy.

In the h-version of the finite element method the polynomial order p is fixed and the accuracy of the approximation is controlled by mesh refinement. For a sequence of progressively refined uniform meshes, the error in energy is asymptotically related to the parameter α and the number of degrees of freedom N by:

$$e_E^2 \lesssim \frac{k}{N^{\min(\alpha,p)}} \qquad (7)$$

in which k depends on the mesh, the polynomial order p, the solution domain and α [2].

In the p-version the finite element method the mesh is fixed and the accuracy of approximation is controlled by the polynomial order p. The error in energy is asymptotically related to the parameter α and the number of degrees of freedom by:

$$e_E^2 \leq \frac{C}{N^{2\alpha-\varepsilon}} \qquad (\varepsilon > 0 \text{ arbitrarily}) \tag{8}$$

in which C depends on the mesh, the solution domain, α, and ε [2].

In practical problems α is usually between $\frac{1}{2}$ and 1, hence the rate of convergence in energy is controlled by α, and the p-version has twice the asymptotic rate of convergence of the h-version based on uniform mesh refinement.

We remark that it is possible to achieve rates of convergence in the h-version which are independent of the singularity (dependent only on p) if the meshes are adaptively constructed. An example of adaptively constructed meshes was given in [2]. The sequence of adaptively constructed meshes depends on the norm in which the error is measured. If the h and p versions are properly combined then it is possible to achieve even faster, possibly exponential rates of convergence in energy [2,3].

3. Convergence of the root-mean-square measure of stress

We define the root-mean-square error in stress for plane elastic problems as:

$$e_\sigma = \sqrt{\frac{1}{A} \int_A [(\sigma_{11}-\tilde{\sigma}_{11})^2 + (\sigma_{22}-\tilde{\sigma}_{22})^2 + (\sigma_{12}-\tilde{\sigma}_{12})^2]\, dA} \tag{9}$$

in which σ_{11}, σ_{22}, σ_{12} represent the normal and shear stress components corresponding to the exact solution of the problem $\underline{u}$; $\tilde{\sigma}_{11}$, $\tilde{\sigma}_{22}$, $\tilde{\sigma}_{12}$ represent the normal and shear stress components corresponding to the finite element approximation to $\underline{u}$, denoted by $\underline{u}_{FE}$.

Referring to eq's (2) and (3) we can write:

$$\underline{\sigma} - \underline{\sigma}_{FE} = [E][D](\underline{u}-\underline{u}_{FE}) \tag{10}$$

and:

$$e_\sigma^{\ 2} = \frac{1}{A} \int_A \left\{[D](\underline{u}-\underline{u}_{FE})\right\}^T [E]^T[E] \left\{[D](\underline{u}-\underline{u}_{FE})\right\} dA \tag{11}$$

Comparing $e_\sigma^{\ 2}$ with $e_E^{\ 2}$, (eq's. 5,6 and 9) we see that the only differences are that in $e_\sigma^{\ 2}$ we have $[E]^T[E]$ instead of $[E]$ and the integral expression is divided by the area of the solution domain instead of 1/2. We shall now show that this will not affect the asymptotic rate of convergence:

Because [E] is symmetric and positive definite, it can be written as:

$$[E] = [Q]^T[\Lambda][Q] \tag{12}$$

where [Q] is the matrix of normalized eigenvectors of [E] (hence $[Q]^T[Q] = [I]$) and $[\Lambda]$ is the diagonal matrix of the eigenvalues of [E]. Thus we can write:

$$[E]^T[E] = [E][E] = [Q]^T[\Lambda]^2[Q] \tag{13}$$

Denoting:

$$\underline{w} = [Q][D](\underline{u}-\underline{u}_{FE}) \tag{14}$$

We have:

$$e_\sigma^{\ 2} = \frac{1}{A} \int_A \underline{w}^T [\Lambda]^2 \underline{w} \, dA \tag{15}$$

$$e_E^{\ 2} = \frac{1}{2} \int_A \underline{w}^T [\Lambda] \, \underline{w} \, dA \tag{16}$$

Because the eigenvalues of [E] are all positive, the integrands in eq's (15) and (16) are both positive and, furthermore;

$$\lambda_{min} \, \underline{w}^T [\Lambda] \, \underline{w} \le \underline{w}^T [\Lambda]^2 \, \underline{w} \le \lambda_{max} \, \underline{w}^T [\Lambda] \, \underline{w} \tag{17}$$

where λ_{max} and λ_{min} are the largest and smallest eigenvalues of [E]. Consequently:

$$\sqrt{2\lambda_{min}} \, e_E \le \sqrt{A} \, e_\sigma \le \sqrt{2\lambda_{max}} \, e_E \tag{18}$$

λ_{min} and λ_{max} are, of course, independent of the finite element mesh and the polynomial order. In the case of isotropic materials the eigenvalues of [E] are functions of the modulus of elasticity E and Poisson's ratio ν only.* Thus we conclude that as the number of degrees of freedom

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*In the case of isotropic plane strain:

$$\lambda_{max} = \frac{E}{(1+\nu)(1-2\nu)} \quad ; \quad \lambda_{min} = \frac{E}{2(1+\nu)}$$

In the case of isotropic plane stress:

$$\lambda_{max} = \frac{E}{1-\nu} \quad ; \quad \lambda_{min} = \frac{E}{2(1+\nu)}$$

are increased either through mesh refinement or increased polynomial order, or both, e_σ will approach zero at the same rate as e_E.

Since we are interested in the relative error, we define the root-mean-square measure of stress as:

$$S(\underline{\sigma}) = \sqrt{\frac{1}{A} \int_A (\sigma_{11}^2 + \sigma_{22}^2 + \sigma_{12}^2)\, dA} \tag{19}$$

In terms of displacements $\underline{u}$ S can be written as:

$$S(\underline{u}) = \sqrt{\frac{1}{A} \int_A ([D]\underline{u})^T [Q]^T [\Lambda]^2 [Q]([D]\underline{u})\, dA} \tag{20}$$

which is analogous to the energy norm $\sqrt{U(\underline{u})}$. Thus we can show that

$$\sqrt{2\lambda_{min}}\, \sqrt{U(\underline{u})} \leq \sqrt{A}\, S\, (\underline{u}) \leq \sqrt{2\lambda_{max}}\, \sqrt{U(\underline{u})} \tag{21}$$

Let us define the relative error in stress as:

$$(e_r)_\sigma = \frac{S(\underline{u}-\underline{u}_{FE})}{S(\underline{u})} \tag{22}$$

and the relative error in strain energy norm as:

$$(e_r)_E = \sqrt{\frac{U(\underline{u}-\underline{u}_{FE})}{U(\underline{u})}} \tag{23}$$

In view of these definitions and eq's (18,21), we can write:

$$\sqrt{\frac{\lambda_{min}}{\lambda_{max}}} \le \frac{(e_r)_\sigma}{(e_r)_E} \le \sqrt{\frac{\lambda_{max}}{\lambda_{min}}} \qquad (24)$$

Thus the relative error in the root-mean-square measure of stress can be smaller or larger than the relative error in strain energy norm, within the bounds given by inequalities (24). Because the bounds correspond to rather extreme situations, it can be expected that in practical problems the two relative errors will be much closer than the bounds would indicate. In those cases for which the exact solution is not known, it is much easier to compute the error in strain energy norm than the root-mean-square error in stress components therefore in the following discussion we shall be concerned primarily with error in energy norm.

4. Examples

In the following examples plane strain conditions and Poisson's ratio of 0.3 are assumed. Only uniform mesh refinements and uniform p-distributions are considered. The solutions were obtained by means of COMET-X, an experimental computer code which implements the p-version of the finite element method [4].

4.1 Cantilever beam

The short cantilever beam subjected to uniformly distributed lateral load, as shown in the inset of Fig. 2, is our first example. There are two stress singularities which occur at those points where the constrained side intersects the loaded and free sides. Both singularities are characterized by $\alpha = 0.7112$ [cf. eq.(1)]. The exact strain energy estimated by extrapolation is $0.95171\ q^2/E$ where E is the modulus of elasticity. We have chosen this example problem because this or similar details

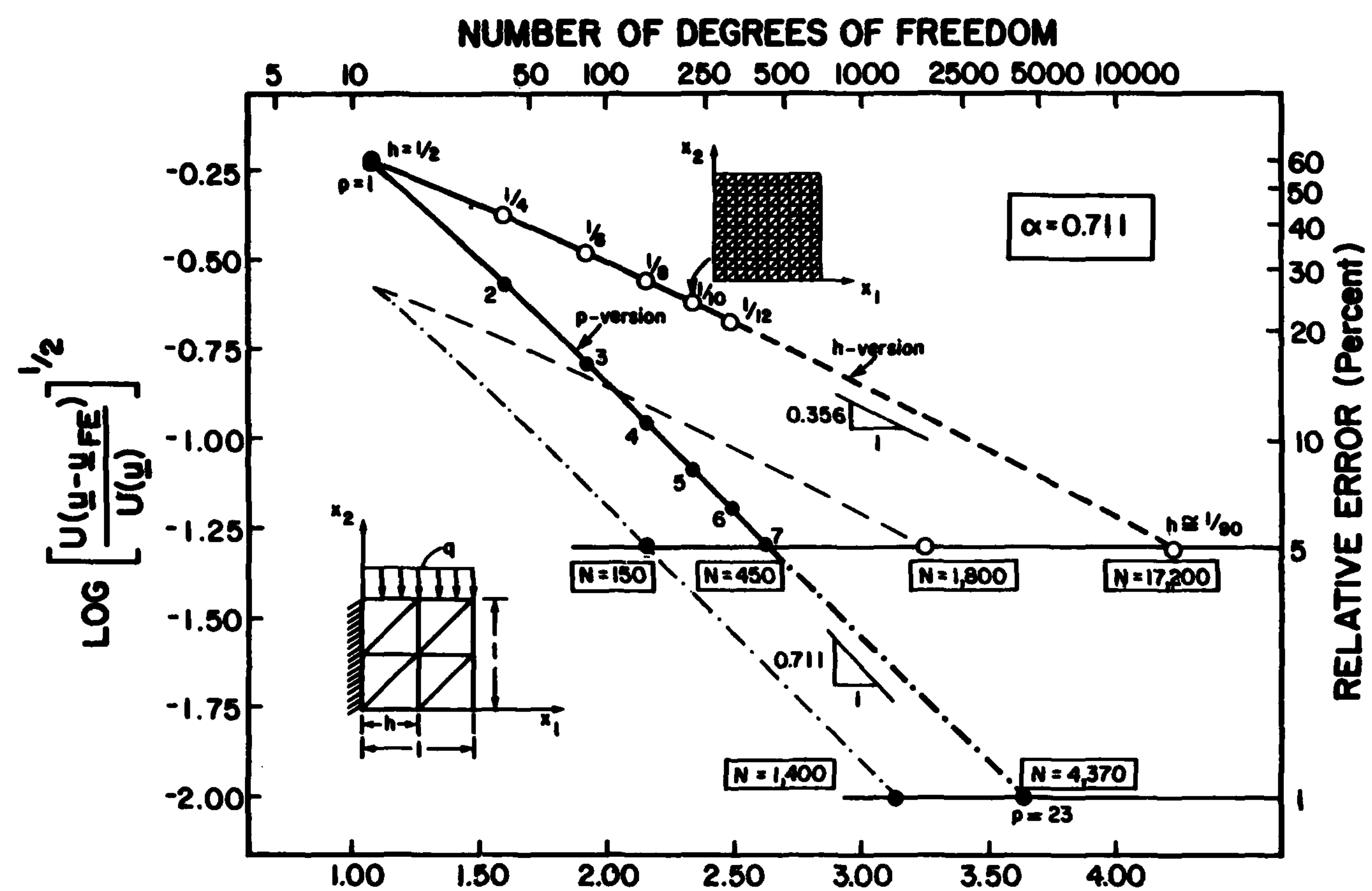

Fig. 2

Relative error in energy norm vs. number of degrees of freedom
Short cantilever beam. Plane strain, ν=0.3

frequently occur in practical stress computations. The relative error in strain energy norm is plotted against the number of degrees of freedom N on log-log scale in Fig. 2. The point corresponding to the highest error (and lowest N) was obtained with $p=1$ and the eight-element mesh shown in the inset. The paths obtained by means of progressive uniform mesh refinement (h-version) and progressively increased p (p-version) are both seen to enter the asymptotic range at low N values i.e. the error substantially follows the theoretically established bounds given by eq's (7,8). The exponent of N in eq's (7,8) appears as the slope on the log-log scale. The light broken lines indicate the lower bounds of the error in the root-mean-square measure of stress on the basis of inequality (24). The upper bounds are not shown.

In the current practice of stress analysis the type of finite element and mesh are chosen by the analyst on the basis of experience. If a survey were conducted, in which given this simple problem, experienced analysts were asked to design a mesh and choose a p within the capabilities of state-of-the-art finite element stress analysis technology, which would guarantee 5 percent relative error in the root-mean-square measure of stress, most would probably say that the 8-element mesh with $p=2$ or the 200-element mesh with $p=1$, shown in insets of Fig. 2, would be sufficient. In fact, Fig. 2 shows that the relative errors would be greater than 25 percent.

We see from the figure that in the p-version the 8-element mesh with $p=7$ or 450 degrees of freedom would be needed for reducing the relative error to 5 percent. In the h-version, with $p=1$, the element size h would have to be approximately 1/90 and the number of degrees

of freedom would be approximately 17,200. If we assume that the variable cost of computation is proportional to the square of the number of degrees of freedom, then the cost of computation would be 126 times greater in the p-version and 6,100 times greater in the h-version than originally estimated. We believe that, relying purely on experience, none of the analysts in our hypothetical survey would have come close to estimating the levels of effort required for obtaining 5 percent relative error in stresses which would guarantee only one significant digit. The situation would have been much worse if one asked for one percent relative error, which does not yet guarantee two significant digits: Fig. 2 indicates that one percent relative error in the root-mean-square measure of stress simply cannot be achieved with state-of-the-art finite element technology.

If on the other hand our goal were to compute the average displacement of the top fiber of the cantilever rather than the root-mean-square measure of stress, the problem would be much easier to solve. In this case the relative error, $(e_r)_D$ is defined as:

$$(e_r)_D = \frac{\left|\int_0^1 (u_2 - \tilde{u}_2)\,dx_1\right|_{x_2=1}}{\left|\int_0^1 u_2\,dx_1\right|_{x_2=1}} \tag{25}$$

Recognizing that the strain energy can be computed from:

$$U(\underline{u}) = \frac{1}{2}\,q\int_0^1 u_2(x_1, x_2=1)\,dx_1 \tag{26}$$

We find that $(e_r)_D = (e_r)_E^2$. This means, for example, that 5 percent relative error in average displacement corresponds to 22.4 percent relative error in strain energy norm. The expected responses to our hypothetical survey would have given close estimates of the required effort for this case.

4.2 Double edge cracked panel

The double edge cracked panel shown in Fig. 3 is our second example. Because of symmetry only one quarter of the domain needs to be analyzed. The estimated exact strain energy for the quarter domain is 0.73410 σ^2/E. The 8-element mesh shown in Fig. 3 is the same as in the previous example. Stress singularities occur at the crack tips, the singularities are characterized by $\alpha=0.5$. The relative error in strain energy norm is plotted against the number of degrees of freedom N on log-log scale in Fig. 4. The light broken lines indicate the lower bounds of the relative error in the root-mean-square measure of stress for the h- and p-versions.

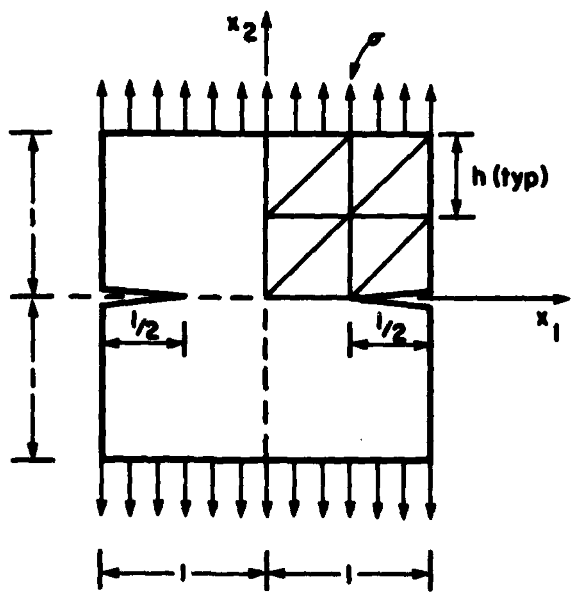

Fig. 3
Double edge cracked square panel

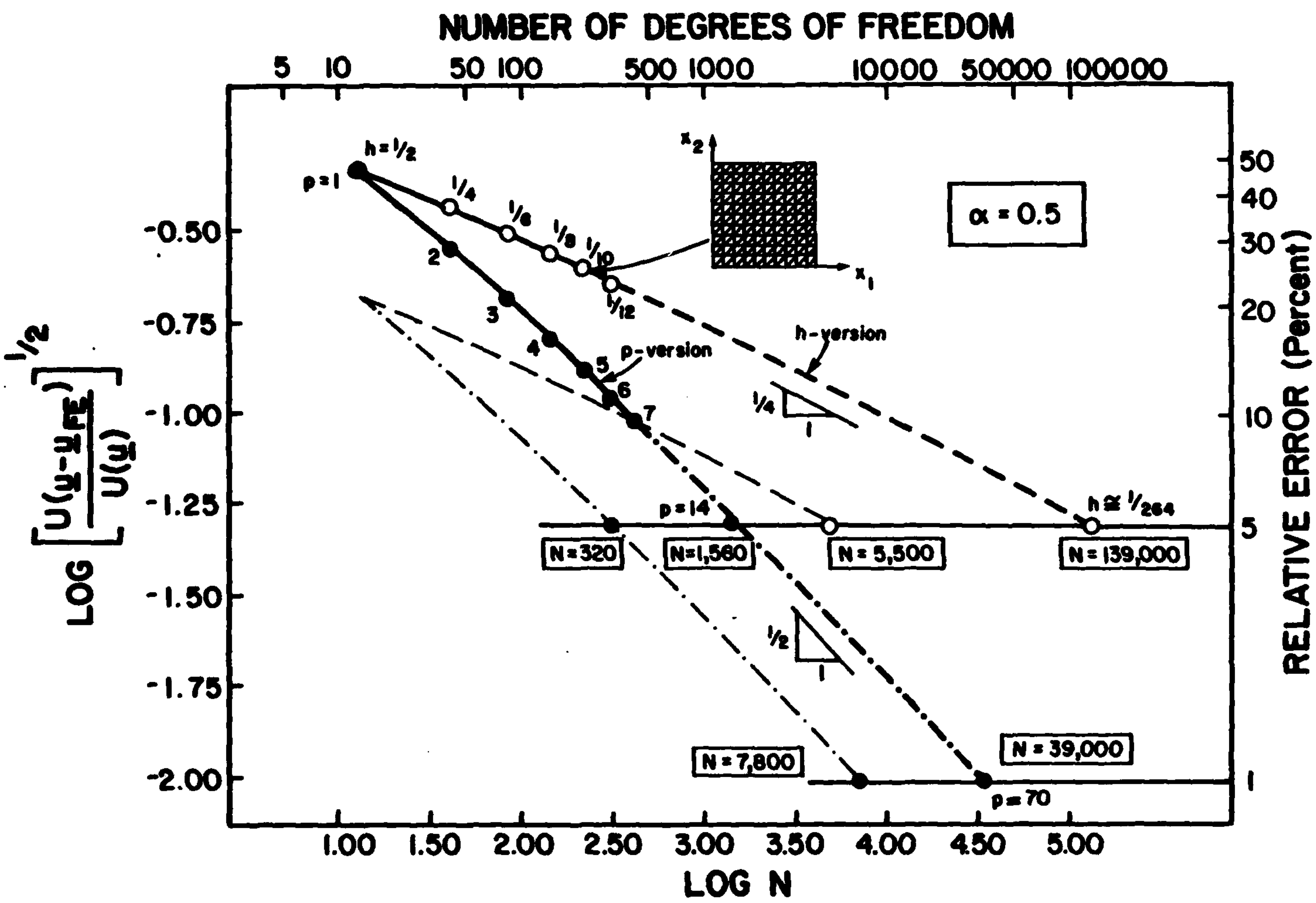

Fig. 4

Relative error in energy norm vs. number of degrees of freedom
Double edge cracked square panel. Plane strain, ν=0.3

Fig. 4 shows that in this case 5 percent relative error in strain energy norm is well beyond the capabilities of state-of-the-art finite element technology: the p-version would require 1560 degrees of freedom with p=14; the h-version 139,000 degrees of freedom, the element size h being approximately 1/264.

5. Convergence of the stress intensity factor

With increasing importance and frequency, determination of stress intensity factors is the main goal of finite element computations in structural stress analysis. When only one mode of fracture is of interest then the stress intensity factor is most efficiently computed from the strain energy release rate. The strain energy release rate G is defined as the absolute value of the rate of change of strain energy with respect to crack length, a:

$$G = \left|\frac{\partial U}{\partial a}\right| \tag{27}$$

In view of equations (6,7,8) the rate of convergence is the same as the rate of convergence of strain energy rather than that of the strain energy norm or, equivalently, the root-mean-square measure of stress. Consequently the slope of the error in G ($G-G_{FE}$) is twice the slope of e_E, or e_σ, when $G-G_{FE}$ is plotted against the number of degrees of freedom on log-log scale (Fig. 5). We remark that the rate of convergence shown in Fig. 5 for the h-version can be realized only if the procedure for obtaining $\frac{\partial U}{\partial a}$ is properly designed and implemented.

The value of G can be readily determined to about four significant digits by means of extrapolation, utilizing the asymptotic convergence rate. The procedure was described in references [5,6]. In the case

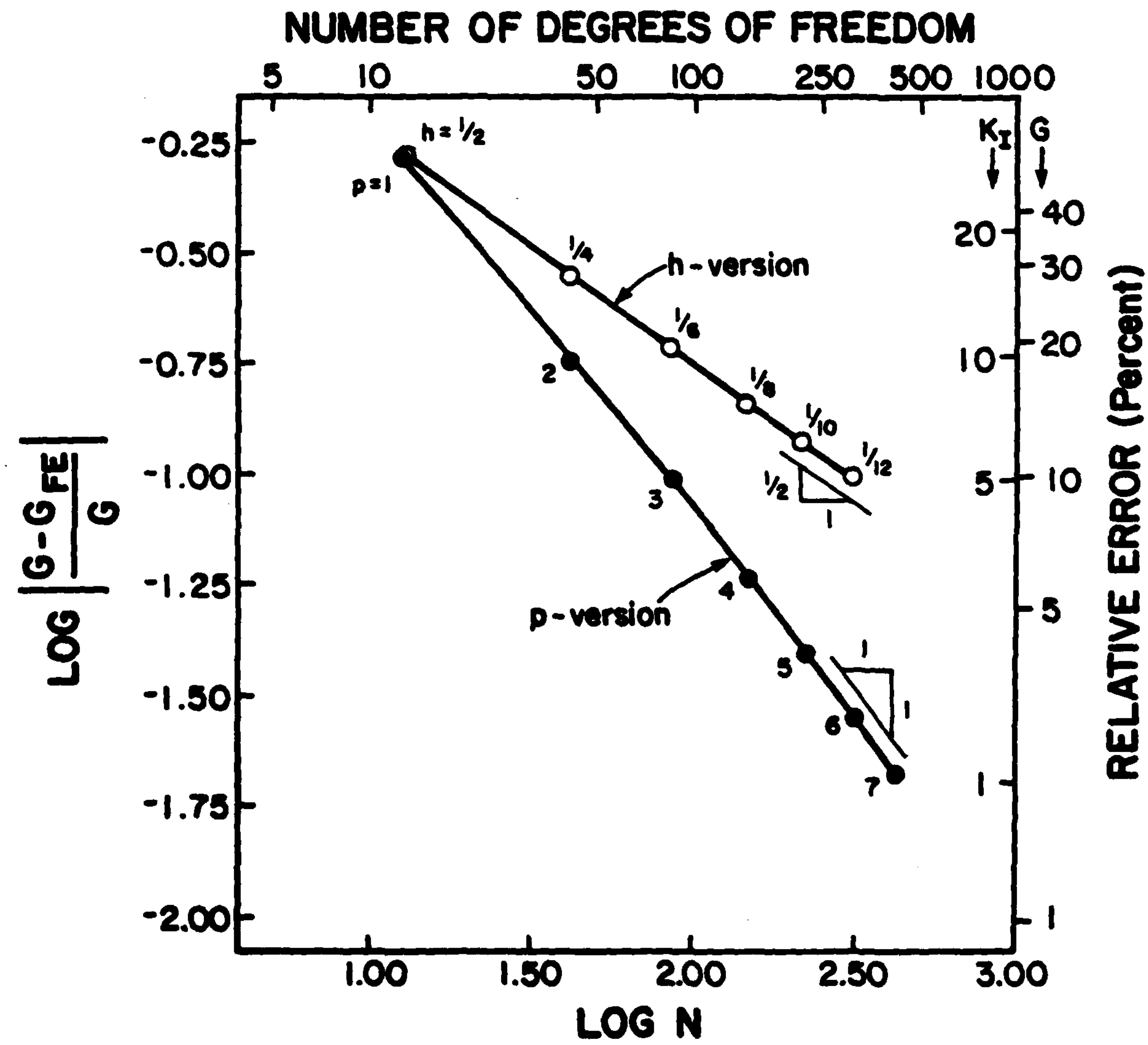

Fig. 5

Relative error in strain energy release rate and K_I vs. N
Double edge cracked panel. Plane strain, ν=0.3.

of example problem 4.2 the value of G is 2.538 σ^2/E. The corresponding value of the stress intensity factor K is computed from the well known relationship:

$$K = \sqrt{\frac{GE}{1-\nu^2}} \tag{28}$$

for plane strain (and $K = \sqrt{GE}$ for plane stress) in which E is the modulus of elasticity, ν is Poisson's ratio. Because K is proportional to the square root of G, the relative error in K is a simple function of the relative error in G. The relationships between relative errors in K_I and G and number of degrees of freedom are shown on the basis of example problem 4.2 for the h- and p-versions in Fig. 5. It is seen that K_I can be computed to within 5 percent relative error with very reasonable computational effort by means of the p-version. It can also be computed to 5 percent relative error with the h-version but control of error by mesh refinement is much more difficult.

The computation of K is a much simpler problem than the computation of the average stress as measured by the root-mean-square formula, eq. (19).

When both modes of fracture are present and separation of K_I and K_{II} is necessary the stress intensity factors are determined from the displacement field [7]. This procedure is less precise than the strain energy release rate method. It has been demonstrated, however, that it is possible to estimate K_I, K_{II} to within 2 to 3 percent relative error even with coarse finite element meshes if p is in the range of 6 to 8 [6].

6. Estimation of error

The results indicate that computed data can be and often are outside of the range of relative errors normally expected in engineering computations,

even when extremely fine meshes are used. The value of computations would be greatly enhanced therefore if the appropriate error bounds were reported along with the computed data. It has been proven and demonstrated that reliable estimators of error in various norms can be computed for the h-version [8]. The p-version poses more difficult problems in this area but some progress has been made [9]. Such estimators will give close estimates of error in the general case (i.e. when the exact solution is not known) and will require a relatively small amount of additional computation. It is expected that eventually all finite element computer codes will incorporate error estimators.

Given the current state of the art in finite element technology, which can be characterized briefly as h-version computer codes containing elements with $p=1,2$ and, in some cases, $p=3$; availability of isoparametric elements and mesh generators, error bounds can be obtained indirectly through the design of carefully established benchmark problems. Such benchmark problems must contain the essential features of "real life" problems. In the two examples presented here, the loadings and the corner singularities were the essential features that determined the error of approximation and the requirements for controlling it. Were it not for the corner singularities, the two problems would have been very simple to solve. Singularities can be also induced by loading, sudden changes in boundary conditions and material properties. In the examples presented here the loadings did not induce singularities, but the way in which the loading is applied does have an effect on the relative error associated with a given mesh and p-distribution. Other numerical difficulties occur when Poisson's ratio is close to 0.5 or when a thin plate is

analyzed by a theory that accounts for shear deformation. The benchmark problems must account for all such phenomena.

7. Control of error

We have seen that our ability to control the error of approximation with the expenditure of reasonable amounts of effort depends mainly on the rate of convergence. The rate of convergence depends on the norm in which the error is measured, which is determined by the purpose of computation and the strategy by which the number of degrees of freedom is increased. Here we have demonstrated two strategies only: uniform mesh refinement and uniform increase in polynomial order.

When the goal of computation is to determine the state of stress at specific points or the maximum value of one or more stress components, the rate of convergence is even slower than in the case when stress approximation in the least squares sense is of interest, unless special techniques are used for extracting stress values from the computed data such as, for example, computing stresses at properly chosen points on a regular mesh.

The example problems presented here are not the worst possible cases from the point of view of error control: In plane elasticity, when the material is homogeneous and isotropic, α can be as low as 1/4 and when the material is nonhomogeneous, α can be an arbitrary small number greater than zero. The computational effort required for effecting meaningful reduction of error is an extremely sensitive function of α.

In the examples presented we have considered uniform mesh refinement only. This indicates a larger number of degrees of freedom than normally used in practice since most analysts will grade the mesh so

that the element size is progressively reduced toward the singularity. The size of the elements at the singularity is therefore a more realistic indicator of the required computational effort than the number of degrees of freedom shown in Figures 2, 4, 5. Unless the sequence of mesh refinement is properly designed, however, the asymptotic rate of convergence, i.e. the slope on the error vs. N diagram will not change. Examples of properly designed sequences of mesh refinement are available in [2,3,8].

Let us consider the effort required for halving the relative error in a given case, assuming that the error is controlled by corner singularities only. The variable computer resource requirement or cost C is proportional to N^{β}.

$$C(e_r) \sim N^{\beta} \tag{29}$$

in which N is the number of degrees of freedom and C is, of course, the function of the relative error e_r. If the slope of the relative error vs. N curve on log-log scale is μ then the cost of halving the relative error is:

$$C(\tfrac{1}{2}\, e_r) = 2^{\beta/\mu} C(e_r) \tag{30}$$

In most practical problems β is close to two for both the h- and p-versions. Thus, when $\mu=1$ the cost of halving the relative error quadruples. In practical problems, however, μ is generally less than 1. In the case of the short cantilever, for example, $\mu=0.356$ when the conventional h-version is used and the goal of computation is to

approximate stresses in the root mean square sense, and μ=0.711 when the goal is to compute the average displacement of the top fiber. The corresponding figures for the p-version are exactly twice as large (0.711 and 1.422).

Let us estimate the variable computer resource requirements when the purpose of computation is to approximate stresses such that the acceptable relative error in the root-mean-square measure of stress is 5 percent. We assume that an initial approximate solution is available, which represents our first 'guess' of what the mesh and p should be. Associated with this initial solution is a number of degrees of freedom N_o; a variable computer resource requirement C_o and a relative error $(e_r)_o$ which we assume to be greater than 5 percent. For the purposes of the following discussion we further assume that C is proportional to N^β, as stated in q.(29), with β=2, and the initial solution is in the asymptotic range, i.e. N_o is sufficiently large so that the equal sign replaces the 'less or equal' sign in eq's(7,8).

Depending on the strength of the geometric and loading singularities and the strategy by which the number of degrees of freedom are increased in order to reduce $(e_r)_o$ to 5 percent, a convergence rate is realized. Unless properly designed mesh refinement sequences and p-distributions are employed, which is well beyond the current state of the art in finite element analysis, μ is independent of N.

Based on equations (7,8 and 29) we can compute the ratio C/C_o as a function of $(e_r)_o$ in which C is the variable computer resource requirement for reducing the error from $(e_r)_o$ to 5 percent. The results are shown in Fig. 6. We see that C/C_o is an extremely sensitive function of μ.

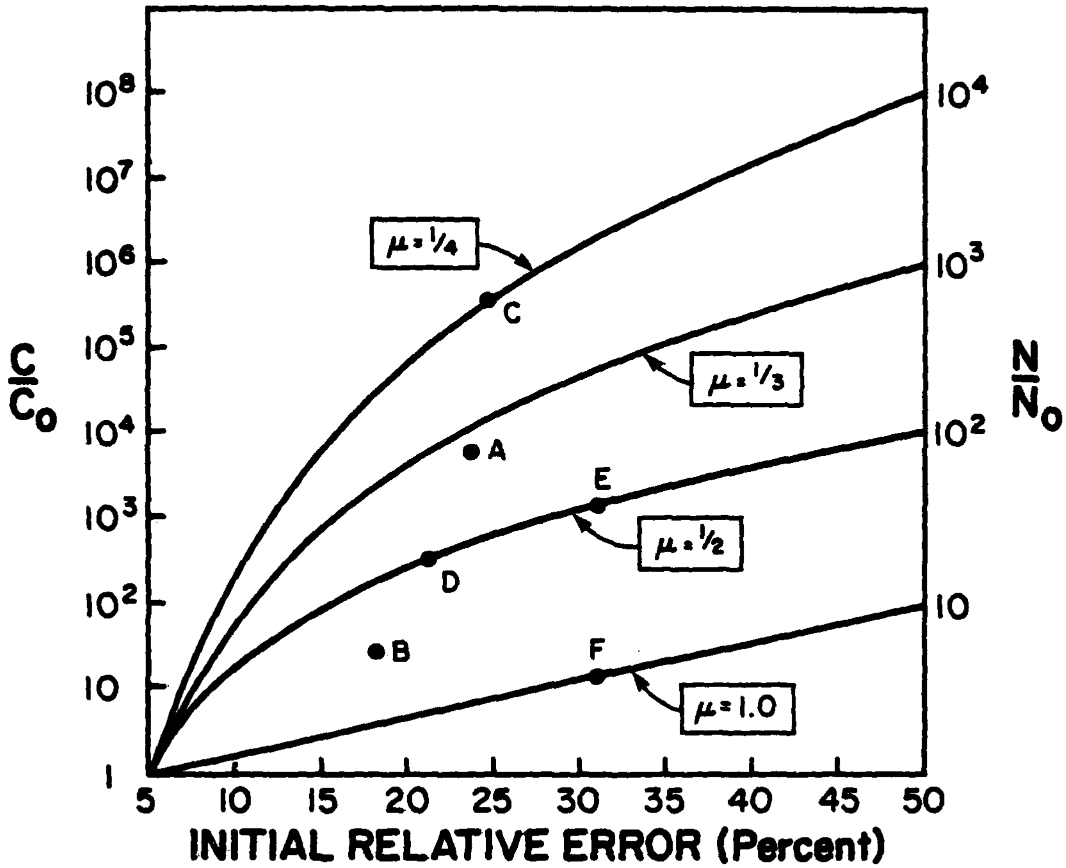

Fig. 6

Effect of the rate of convergence (μ) on the variable computer resource requirements (C) when an initial relative error in the range of 5 to 50 percent is to be reduced to 5 percent, assuming $\beta=2$.

Legend:

Short cantilever:

A: $h=1/10$ $p=1$, $N_o=220$, h-version, RMS stress.
B: $h=1/2$ $p=3$, $N_o=84$, p-version, RMS stress.

Double edge cracked panel:

C: $h=1/10$ $p=1$, $N_o=225$, h-version, RMS stress.
D: $h=1/2$ $p=3$, $N_o=87$, p-version, RMS stress.
E: $h=1/2$ $p=1$, $N_o=13$, h-version, K_I
F: $h=1/2$ $p=1$, $N_o=13$, p-version, K_I

Points A and B in Fig. 6 represent initial solutions for the short cantilever beam, the details of which are given in the legend. There are corresonding points in Fig. 2. We see that it is not feasible to reduce $(e_r)_o$ to five percent by means of uniform mesh refinement and p=1.

Points C and D represent initial solutions for the double edge cracked panel. We see that in this case neither uniform mesh refinement nor uniform increases in p make it possible to reduce the error in e_σ to 5 percent in practice. Fortunately it is much easier to control the error when the goal is to compute the stress intensity factor K_I. Points E and F represent initial solutions for K_I for the h and p-versions, respectively. Although the relative error of the initial solution is large, N_o is small and therefore it is feasible to compute K_I to within 5 percent relative error with both the h- and p versions.

8. Summary and conclusions

We have considered the problem of controlling the error in the root-mean-square measure of stress in plane elasticity. It was shown that for most problems of practical importance it is not feasible to exercise substantive control over the error in this norm by conventional finite element technology. The p-version will extend the range of problems for which control in this norm is feasible and will substantially reduce the difficulties associated with controlling the relative error in stress intensity factors.

Our ability to control the error of approximation in engineering practice depends on the following factors:

(1) the purpose of computation, which determines the norm(s) in which the error is measured (such as displacements, stresses, stress intensity factors, natural frequencies, etc.) and the acceptable relative error;

(2) the rate of convergence, which depends on the strength of geometric and loading singularities and the available strategies by which the degrees of freedom can be increased;

(3) the required range of error reduction and the current number of degrees of freedom (N_o) which depend on the finite element mesh, the polynomial order, the domain and the loading;

(4) the availability of reliable and close estimates of error in the appropriate norm at any given stage of computation.

The development of error estimators is currently in mature stages of research. Eventually they will be generally available in finite element codes used in engineering practice. Until such time, carefully designed benchmark problems provide means for the assessment and assurance of quality in finite element analysis.

A finite element computer code cannot be accepted or certified on the basis that it provides very close approximations when tested against problems for which the exact solution is known because such problems generally lack the essential details that determine the error in practical applications. In any case, *the intrinsic properties of a computer code do not guarantee that the results in a given application will conform to some general standard.* Proper use of the code, which depends on the problem to be solved, the goal(s) of computation as well as the capabilities of the code, is essential. Therefore the design of benchmark problems must rest on the same three considerations.

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